

# Chaotic Behaviour Revisited

by Gérard A. Langlet

## What is Chaotic Behaviour?

Chaotic behaviour occurs when one is unable to predict what will happen at some distance from here or from now. It has long been known (Poincaré, Lorenz with the butterfly-effect) that a small variation in the initial conditions may lead to huge differences in the behaviour of dynamical systems, which, in general, are described by differential equations as a function of time.

Hundreds of books and papers are devoted to chaos which is supposed to appear even when one iterates very simple nonlinear equations such as the "logistic equation", proposed by Verhulst more than 130 years ago in order to model and explain population ratio (alternate growth and decay) in ecological systems.

If  $X$  is a variable which may vary in the interval  $\{0,1\}$ , then Verhulst's **nonlinear** formula:

$$X_{n+1} = 4 X_n (1-X_n)$$

will give the next population ratio at step  $n+1$  if one knows the ratio at step  $n$ .

Although this formula is completely deterministic (ALL equations are deterministic), the succession of  $X_i, X_{i+1} \dots X_{i+p}$  (terms of a series) exhibits a "positive Lyapunov exponent", **proof** of its chaotic behaviour.

When the behaviour is not chaotic, this exponent is simply 0.

It is clear that iterations of linear functions may NEVER lead one to observe any chaotic behaviour.

## *Hic jacet Chaos* (a non-syllogistic mathematical proof)

So, let us consider the iterations of one of the most simple **linear** formulas one can imagine:

$$\omega_{n+1} = 2\omega_n \quad \text{with } \omega \text{ an angle (e.g. expressed in radians).}$$

By no means would the iterations of such a formula lead to chaos.

As an example, if  $\omega_0$  is 1 radian, one may immediately predict that  $\omega_N$  shall exactly equal  $2^N$  radians.

Then, in order to obtain a variation in the closed interval  $\{0,1\}$ , let us choose variable  $X$  as  $\sin^2\omega$  and replace the series involving powers of 2 by a series in  $X$ .

Any value of  $X$  is still predictable, as the squared sine of the corresponding  $\omega$ .

For any value of  $X_n = \sin^2\omega_n$ , the next value will be  $X_{n+1} = \sin^2\omega_{n+1}$  i.e.  $\sin^2 2\omega_n$ .

For any value of  $\omega$ , then  $\sin^2 2\omega_n$  may be written as a function of  $\omega_n$  simply as:  $(2 \sin\omega_n \cos\omega_n)^2$  which is equivalent to:  $4 \sin^2\omega_n \cos^2\omega_n$ .

Knowing that  $\sin^2\omega_n$  is  $X_n$ , and that  $\cos^2\omega_n$  is  $1-X_n$ , any undergraduate student finds the "logistic" formula:

$$X_{n+1} = 4 X_n (1-X_n)$$

which, consequently, MAY NOT be chaotic anymore.

*Quod erat demonstrandum.*

## The True Origin of Chaos for Iterated Applications

Successive iterations of the logistic equation are ALWAYS computed ... with computers.

In all computers, precision is limited. (On paper, with a slide rule or a calculator, precision is also limited.)

If any initial value of  $\omega$  is coded with  $B$  bits, every new iteration would require ONE new bit on the left of the internal representation of the old  $\omega$ . Doubling  $\omega_n$  simply appends a new 0 to the right of the previous representation of  $\omega_n$ .

So, the internal representation of  $\omega_{n+1}$  is the same as the one of  $\omega_n$ , with a 1-bit left shift; one can also say that a 0 on the left of  $\omega_n$  is transferred to the right with a 1-position circular shift, in order to produce  $\omega_{n+1}$ .

